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Gravitational four-fermion interaction on the Planck scale

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ABSTRACT

The four-fermion gravitational interaction is induced by torsion, and gets essential on the Planck scale. On this scale, the axial–axial contribution dominates strongly the discussed interaction. The energy–momentum tensor, generated by this contribution, is analyzed, as well as stability of the problem with respect to compression. The trace of this energy–momentum tensor can be negative.

1. The observation that, in the presence of torsion, the interaction of fermions with gravity results in the four-fermion interaction of axial currents, goes back at least to [1].

We start our discussion of the four-fermion gravitational interaction with the analysis of its most general form.

As has been demonstrated in [2], the common action for the gravitational field can be generalized as follows:

$$S_g = -\frac{1}{16\pi G}\int d^4x\,(-e)e^\mu_I e^\nu_J\left(R^{IJ}_{\mu\nu} - \frac{1}{\gamma}\tilde{R}^{IJ}_{\mu\nu}\right), \tag{1}$$

here and below G is the Newton gravitational constant, $I, J = 0, 1, 2, 3$ are internal Lorentz indices, $\mu, \nu = 0, 1, 2, 3$ are space-time indices, e^I_μ is the tetrad field, e is its determinant, and e^μ_I is the object inverse to e^I_μ. The curvature tensor is

$$R^{IJ}_{\mu\nu} = -\partial_\mu \omega^{IJ}{}_\nu + \partial_\nu \omega^{IJ}{}_\mu + \omega^{IK}{}_\mu \omega_K{}^J{}_\nu - \omega^{IK}{}_\nu \omega_K{}^J{}_\mu,$$

here ω^{IJ}_μ is the connection. The first term in (1) is in fact the common action of the gravitational field written in tetrad components.

The second term in (1), that with the dual curvature tensor

$$\tilde{R}^{IJ}_{\mu\nu} = \frac{1}{2}\varepsilon^{IJ}{}_{KL} R^{KL}_{\mu\nu},$$

does not vanish in the presence of spinning particles generating torsion [3].

As to the so-called Barbero–Immirzi parameter γ, its numerical value

$$\gamma = 0.274 \tag{2}$$

was obtained for the first time in [4], as the solution of the "secular" equation

$$\sum_{j=1/2}^{\infty}(2j+1)e^{-2\pi\gamma\sqrt{j(j+1)}} = 1, \tag{3}$$

derived in Ref. [4].

Thus derived effective four-fermion interaction of axial currents is [3]:

$$S_A = \frac{3}{2}\pi\frac{\gamma^2}{\gamma^2+1}G\int d^4x\,(-e)\eta_{IJ}A^I A^J. \tag{4}$$

Here and below $\eta_{IJ} = \mathrm{diag}(1, -1, -1, -1)$, A^I is the net fermion axial current:

$$A^I = \sum_a A^I_a = \sum_a \bar{\psi}_a\gamma^5\gamma^I\psi_a, \tag{5}$$

the sum extends over all sorts of fermions with spin 1/2. This result (4) corresponds (up to a factor) to that derived long ago in [1].

In the absence of the pseudoscalar term in gravitational action (1), i.e. for $\gamma \to \infty$, expression (4) simplifies to

$$S_A^\infty = \frac{3}{2}\pi G\int d^4x\,(-e)\eta_{IJ}A^I A^J. \tag{6}$$

This effective gravitational four-fermion interaction in the limit $\gamma \to \infty$ was derived previously in [5] (when comparing the corresponding result from [5] with (6), one should keep in mind that the convention $\eta_{IJ} = \mathrm{diag}(-1, 1, 1, 1)$, used in [5], differs in sign from ours).

Two other contributions to the effective gravitational four-fermion interaction (4) arise as follows. The common action for fermions in gravitational field

$$S_f = \int d^4x\,(-e)\frac{1}{2}\left[\bar{\psi}\gamma^I e^\mu_I i\nabla_\mu\psi - i\overline{\nabla_\mu\psi}\gamma^I e^\mu_I\psi\right] \tag{7}$$

can be generalized in the following way (see [6]):

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$$S_f = \int d^4x\,(-e)\frac{1}{2}\big[(1-i\alpha)\bar\psi\gamma^I e_I^\mu i\nabla_\mu\psi - (1+i\alpha)i\overline{\nabla_\mu\psi}\gamma^I e_I^\mu\psi\big] \tag{8}$$

here

$$\nabla_\mu = \partial_\mu - \frac{1}{4}\omega_\mu^{IJ}\gamma_I\gamma_J, \qquad [\nabla_\mu, \nabla_\nu] = \frac{1}{4}R_{\mu\nu}^{IJ}\gamma_I\gamma_J.$$

The real constant α introduced in (8) is of no consequence, generating only a total derivative, if the theory is torsion free. However, in the presence of torsion this constant gets operative. In particular, as demonstrated in [6], it results in the following generalization of the four-fermion interaction (5):

$$S_{ff} = \frac{3}{2}\pi G\frac{\gamma^2}{\gamma^2+1}\int d^4x\,(-e)\Big[\eta_{IJ}A^IA^J + \frac{\alpha}{\gamma}\eta_{IJ}\big(V^IA^J + A^IV^J\big) - \alpha^2\eta_{IJ}V^IV^J\Big], \tag{9}$$

here V^I is the net fermion vector current:

$$V^I = \sum_a V_a^I = \sum_a \bar\psi_a\gamma^I\psi_a, \tag{10}$$

and this sum, as well as the analogous one (5) for the axial current, extends over all sorts of elementary fermions with spin 1/2; of course, both A_a^I and V_a^I are neutral currents. As to the negative sign at the AA term in the corresponding formula of [6], it is incorrect (I am grateful to A.A. Pomeransky for pointing out this fact to me).

In fact, VA and VV terms in formula (9) are of no real interest for the problem considered. Indeed, as follows from simple dimensional arguments, interaction (9), being proportional to the Newton constant G and to the particle number density squared, gets essential and dominates over the common interactions only at very high densities and temperatures, i.e. on the Planck scale and below it. Under these extreme conditions, the number densities of both fermions and antifermions increase, due to the pair creation, but the total vector current density V^I remains intact. However, the situation with the axial current density A^I is quite different. As distinct from the C-odd vector current V^I, the axial one A^I is C-even. Therefore, fermions and antifermions contribute to A^I with the same sign, so that A^I grows together with density and temperature. Thus, if we confine to the Planckian regime (the only one, where the four-fermion gravitational interaction (9) can be essential), the V-dependent contributions to this interaction can be neglected, and interaction (9) reduces to the axial–axial one (4).

Thus, the V-dependent contributions to the four-fermion gravitational interaction (9), i.e. terms proportional to α and α^2, are negligibly small everywhere: above the Planck scale they are small together with the purely axial AA interaction, on the Planck scale and below it they are small as compared to the AA interaction.

2. Let us consider now the energy–momentum tensor (EMT) $T_{\mu\nu}$ generated by action (4). Therein, the scalar product $\eta_{IJ}A^IA^J$ has no explicit dependence at all either on the metric tensor, or on its derivatives. The metric tensor enters action S_A only via $-e = \sqrt{-g}$, so that

$$S_A = \frac{3}{2}\pi\frac{\gamma^2}{\gamma^2+1}G\int d^4x\sqrt{-g}\,\eta_{IJ}A^IA^J. \tag{11}$$

Since S_A is independent of the derivatives of metric tensor, the corresponding EMT is given by relation

$$\frac{1}{2}\sqrt{-g}\,T_{\mu\nu} = \frac{\delta}{\delta g_{\mu\nu}}S_A. \tag{12}$$

Now, with identity

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}, \tag{13}$$

we arrive at the following expression for the EMT:

$$T_{\mu\nu} = -\frac{3\pi}{2}\frac{\gamma^2}{\gamma^2+1}g_{\mu\nu}\eta_{IJ}A^IA^J,$$

or, in the tetrad components,

$$T_{MN} = -\frac{3\pi}{2}\frac{\gamma^2}{\gamma^2+1}\eta_{MN}\eta_{IJ}A^IA^J. \tag{14}$$

We note first of all that this EMT corresponds to the equation of state

$$p = -\varepsilon, \tag{15}$$

here and below $\varepsilon = T_{00}$ is the energy density, and $p = T_{11} = T_{22} = T_{33}$ is the pressure.

Let us analyze this expression for the interaction of two ultrarelativistic fermions (labeled a and b) in their center-of-mass system.

The axial current of fermion a (both of particle and antiparticle!) is

$$\begin{aligned} A_a^I &= \frac{1}{4E^2}\phi_a^\dagger\big\{E\boldsymbol\sigma_a(\mathbf p'+\mathbf p), \big(E^2-(\mathbf p'\mathbf p)\big)\boldsymbol\sigma_a + \mathbf p'(\boldsymbol\sigma_a\mathbf p) + \mathbf p(\boldsymbol\sigma_a\mathbf p') - i[\mathbf p'\times\mathbf p]\big\}\phi_a \\ &= \frac{1}{4}\phi_a^\dagger\big\{\boldsymbol\sigma_a(\mathbf n'+\mathbf n), \big(1-(\mathbf n'\mathbf n)\big)\boldsymbol\sigma_a + \mathbf n'(\boldsymbol\sigma_a\mathbf n) + \mathbf n(\boldsymbol\sigma_a\mathbf n') - i[\mathbf n'\times\mathbf n]\big\}\phi_a, \end{aligned} \tag{16}$$

here E is the energy of fermion a, $\mathbf n$ and $\mathbf n'$ are the unit vectors of its initial and final momenta $\mathbf p$ and $\mathbf p'$, respectively; of course, under the discussed extreme conditions all fermion masses can be certainly neglected. In the center-of-mass system, the axial current of fermion b is obtained from this expression by changing the signs: $\mathbf n \to -\mathbf n$, $\mathbf n' \to -\mathbf n'$. Then, after averaging over the directions of $\mathbf n$ and $\mathbf n'$, we arrive at the following semiclassical expressions for the nonvanishing components of the energy–momentum tensor, i.e. for the energy density ε and pressure p (for the correspondence between ε, p and EMT components see [7, Section 35]):

$$\begin{aligned} \varepsilon &= -\frac{\pi}{48}\frac{\gamma^2}{\gamma^2+1}G\sum_{a,b}\rho_a\rho_b\big(3-11\langle\boldsymbol\sigma_a\boldsymbol\sigma_b\rangle\big) \\ &= -\frac{\pi}{48}\frac{\gamma^2}{\gamma^2+1}G\rho^2(3-11\zeta), \end{aligned} \tag{17}$$

$$\begin{aligned} p &= \frac{\pi}{48}\frac{\gamma^2}{\gamma^2+1}G\sum_{a,b}\rho_a\rho_b\big(3-11\langle\boldsymbol\sigma_a\boldsymbol\sigma_b\rangle\big) \\ &= \frac{\pi}{48}\frac{\gamma^2}{\gamma^2+1}G\rho^2(3-11\zeta), \end{aligned} \tag{18}$$

here and below ρ_a and ρ_b are the number densities of the corresponding sorts of fermions and antifermions, $\rho = \sum_a\rho_a$ is the total density of fermions and antifermions, the summation $\sum_{a,b}$ is performed over all sorts of fermions and antifermions; $\zeta = \langle\boldsymbol\sigma_a\boldsymbol\sigma_b\rangle$ is the average value of the product of corresponding $\boldsymbol\sigma$-matrices, presumably universal for any a and b. Since the number of sorts of fermions and antifermions is large, one can neglect here for numerical reasons the contributions of exchange and annihilation diagrams, as well as the fact that if $\boldsymbol\sigma_a$ and $\boldsymbol\sigma_b$ refer to the same particle, $\langle \boldsymbol{\sigma}_a \boldsymbol{\sigma}_b \rangle = 3$. The parameter ζ, just by its physical meaning, in principle can vary in the interval from 0 (which corresponds to the complete thermal incoherence or to the antiferromagnetic ordering) to 1 (which corresponds to the complete ferromagnetic ordering).

Let us note that, according to (17), the contribution of the gravitational spin–spin interaction to energy density is positive, i.e. the discussed interaction is repulsive for fermions with aligned spins. This our conclusion agrees with that made long ago in [5].

To simplify the further discussion, we will confine to the region somewhat below the Planck scale, so that one can neglect effects due to the common fermionic EMT, originating from the Dirac Lagrangian and linear in the particle density ρ.

3. A reasonable dimensional estimate for the temperature τ of the discussed medium is

$$\tau \sim \rho^{1/3} \sim m_{\mathrm{Pl}} \tag{19}$$

(here and below m_{Pl} is the Planck mass). This temperature is roughly on the same order of magnitude as the energy scale ω of the discussed interaction

$$\omega \sim G\rho \sim m_{\mathrm{Pl}}. \tag{20}$$

Numerically, however, τ and ω can differ essentially. Both options, $\tau > \omega$ and $\tau < \omega$, are conceivable.

If the temperature is sufficiently high, $\tau \gg \omega$, it destroys the spin–spin correlations in formulas (17) and (18). In the opposite limit, when $\tau \ll \omega$, the energy density (17) is minimized by the antiferromagnetic spin ordering. Thus, in both these limiting cases the energy density and pressure simplify to

$$\varepsilon = -\frac{\pi}{16}\frac{\gamma^2}{\gamma^2+1}G\rho^2, \tag{21}$$

$$p = \frac{\pi}{16}\frac{\gamma^2}{\gamma^2+1}G\rho^2. \tag{22}$$

The energy density ε, being negative and proportional to ρ^2, decreases with the growth of ρ. On the other hand, the common positive pressure p grows together with ρ. Both these effects result in the compression of the fermionic matter, and thus make the discussed system unstable.

A curious phenomenon can be possible if initially the temperature is sufficiently small, $\tau < \omega$, so that Eqs. (21), (22) hold. Then the matter starts compressing, its temperature grows, and the correlator $\zeta = \langle \boldsymbol{\sigma}_a \boldsymbol{\sigma}_b \rangle$ arises. When (and if) ζ exceeds its critical value $\zeta_{cr} = 3/11$, the compression changes to expansion. Thus, we arrive in this case at the big bounce situation.

4. The last remark is as follows. It is well known that for a system of point-like particles with the electromagnetic interaction among them, the trace T^μ_μ of its EMT satisfies the condition

$$T^\mu_\mu = g_{\mu\nu}T^{\mu\nu} \geqslant 0. \tag{23}$$

The assumption is usually made that this condition is valid also for other interactions existent in Nature (concerning this see [7, Section 34]).

However, in our problem the trace

$$T^\mu_\mu = \varepsilon - 3p = -\frac{\pi}{12}\frac{\gamma^2}{\gamma^2+1}G\rho^2(3-11\zeta) \tag{24}$$

is negative if $\zeta < 3/11$.

This feature of our problem is closely related to the fact that the system under discussion is unstable with respect to compression for $\zeta < 3/11$.

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